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A G-family of quandles and handlebody-knots

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We introduce the notion of a G-family of quandles and use it to construct invariants for handlebody-knots. Our invariant can detect the chiralities of some handlebody-knots including unknown ones. This is a joint work with Atsushi Ishii, Yeonhee Jang and Kanako Oshiro ([8]).

1 Handlebody-links

A *handlebody-link* is a disjoint union of handlebodies embedded in the 3-sphere S^3. Two handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other. A *spatial graph* is a finite graph embedded in S^3. Two spatial graphs are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other. When a handlebody-link H is a regular neighborhood of a spatial graph K, we say that K *represents* H, or H *is represented by* K. In this paper, a trivalent graph may contain circle components. Then any handlebody-link can be represented by some spatial trivalent graph. A *diagram* of a handlebody-link is a diagram of a spatial trivalent graph which represents the handlebody-link.

An *IH-move* is a local spatial move on spatial trivalent graphs as described in Figure 1, where the replacement is applied in a 3-ball embedded in S^3. Then we have the following theorem.

Theorem 1.1 ([6]). *For spatial trivalent graphs K_1 and K_2, the following are equivalent.*

- *K_1 and K_2 represent an equivalent handlebody-link.*
- *K_1 and K_2 are related by a finite sequence of IH-moves.*
- *Diagrams of K_1 and K_2 are related by a finite sequence of the moves depicted in Figure 2.*

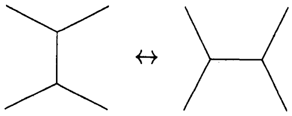

Figure 1:

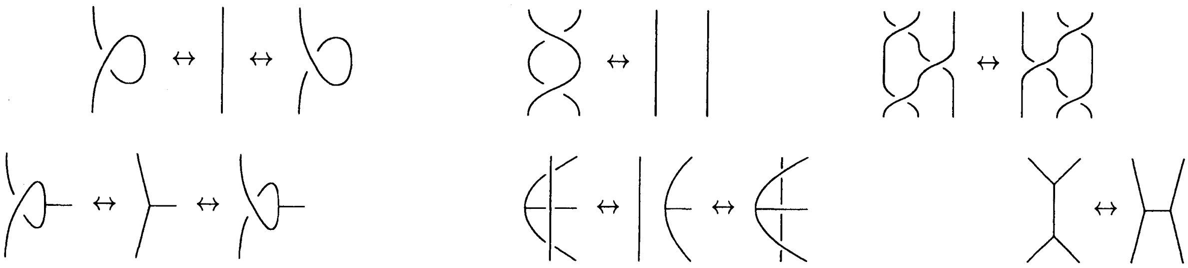

Figure 2:

2 A G-family of quandles

A *quandle* [12, 16] is a non-empty set X with a binary operation $* : X \times X \to X$ satisfying the following axioms.

- For any $x \in X$, $x * x = x$.
- For any $x \in X$, the map $S_x : X \to X$ defined by $S_x(y) = y * x$ is a bijection.
- For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

When we specify the binary operation $*$ of a quandle X, we denote the quandle by the pair $(X, *)$. An *Alexander quandle* $(M, *)$ is a Λ-module M with the binary operation defined by $x * y = tx + (1 - t)y$, where $\Lambda := \mathbb{Z}[t, t^{-1}]$. A *conjugation quandle* $(G, *)$ is a group G with the binary operation defined by $x * y = y^{-1}xy$.

Let G be a group with identity element e. A *G-family of quandles* is a non-empty set X with a family of binary operations $*^g : X \times X \to X$ $(g \in G)$ satisfying the following axioms.

- For any $x \in X$ and any $g \in G$, $x *^g x = x$.
- For any $x, y \in X$ and any $g, h \in G$,
$$x *^{gh} y = (x *^g y) *^h y \text{ and } x *^e y = x.$$
- For any $x, y, z \in X$ and any $g, h \in G$,
$$(x *^g y) *^h z = (x *^h z) *^{h^{-1}gh} (y *^h z).$$

When we specify the family of binary operations $*^g : X \times X \to X$ $(g \in G)$ of a G-family of quandles, we denote the G-family of quandles by the pair $(X, \{*^g\}_{g\in G})$.

Proposition 2.1. *Let G be a group. Let $(X, \{*^g\}_{g\in G})$ be a G-family of quandles.*

(1) For each $g \in G$, the pair $(X, *^g)$ is a quandle.

(2) We define a binary operation $\triangleright : (X \times G) \times (X \times G) \to X \times G$ by

$$(x, g) \triangleright (y, h) = (x *^h y, h^{-1}gh).$$

Then $(X \times G, \triangleright)$ is a quandle.

We call the quandle $(X \times G, *)$ in Proposition 2.1 the *associated quandle* of X.

Example 2.2. (1) Let $(X, *)$ be a quandle. Let $S_x : X \to X$ be the bijection defined by $S_x(y) = y * x$. Let m be a positive integer such that $S_x^m = \mathrm{id}_X$ for any $x \in X$ if such an integer exists. We define the binary operation $*^i : X \times X \to X$ by $x *^i y = S_y^i(x)$. Then X is a $\mathbb{Z}$-family of quandles and a $\mathbb{Z}_m$-family of quandles, where $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$.
(2) Let R be a ring, and G a group with identity element e. Let X be a right $R[G]$-module, where $R[G]$ is the group ring of G over R. We define the binary operation $*^g : X \times X \to X$ by $x *^g y = xg + y(e - g)$. Then X is a G-family of quandles.

3 Colorings

Let D be a diagram of a handlebody-link H. We set an orientation for each edge in D. Then D is a diagram of an oriented spatial trivalent graph K. We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation counterclockwise by $\pi/2$ on the diagram. We denote by $\mathcal{A}(D)$ the set of arcs of D, where an arc is a piece of a curve each of whose endpoints is an undercrossing or a vertex. For an arc α incident to a vertex ω, we define $\epsilon(\alpha; \omega) \in \{1, -1\}$ by

$$\epsilon(\alpha; \omega) = \begin{cases} 1 & \text{if the orientation of } \alpha \text{ points to } \omega, \\ -1 & \text{otherwise.} \end{cases}$$

Let X be a G-family of quandles, and Q the associated quandle of X. Let p_X (resp. p_G) be the projection from Q to X (resp. G). An *X-coloring* of D is a map $C : \mathcal{A}(D) \to Q$ satisfying the following conditions at each crossing χ and each vertex ω of D (see Figure 3).

- Let χ_1, χ_2 and χ_3 be respectively the under-arcs and the over-arc at a crossing χ

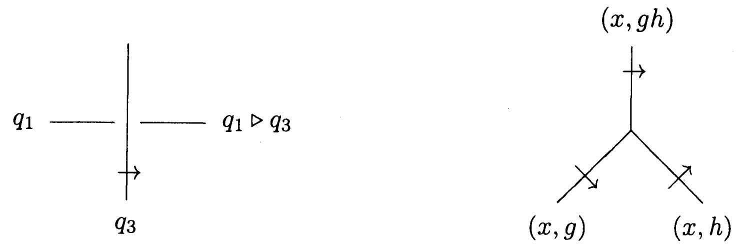

Figure 3:

such that the normal orientation of χ_3 points from χ_1 to χ_2. Then

$$C(\chi_2) = C(\chi_1) \triangleright C(\chi_3).$$

- Let $\omega_1, \omega_2, \omega_3$ be the arcs incident to a vertex ω arranged clockwise around ω. Then

$$(p_X \circ C)(\omega_1) = (p_X \circ C)(\omega_2) = (p_X \circ C)(\omega_3),$$
$$(p_G \circ C)(\omega_1)^{\epsilon(\omega_1;\omega)}(p_G \circ C)(\omega_2)^{\epsilon(\omega_2;\omega)}(p_G \circ C)(\omega_3)^{\epsilon(\omega_3;\omega)} = e.$$

We denote by $\mathrm{Col}_X(D)$ the set of X-colorings of D. For two diagrams D and E which locally differ, we denote by $\mathcal{A}(D, E)$ the set of arcs that D and E share.

Lemma 3.1. *Let X be a G-family of quandles. Let D be a diagram of an oriented spatial trivalent graph. Let E be a diagram obtained by applying one of the R1–R6 moves to the diagram D once, where we choose orientations for E which agree with those for D on $\mathcal{A}(D, E)$. For $C \in \mathrm{Col}_X(D)$, there is a unique X-coloring $C_{D,E} \in \mathrm{Col}_X(E)$ such that $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$.*

Remark 3.2. Let X be a $\mathbb{Z}$-family of quandles or a $\mathbb{Z}_m$-family of quandles defined as in Example 2.2 (2). Then an X-coloring be regarded as an X-coloring defined in [7].

Let X be a G-family of quandles, and Q the associated quandle of X. An *X-set* is a non-empty set Y with a family of maps $*^g : Y \times X \to Y$ satisfying the following axioms, where we note that we use the same symbol $*^g$ as the binary operation of the G-family of quandles.

- For any $y \in Y$, $x \in X$, and any $g, h \in G$,

$$y *^{gh} x = (y *^g x) *^h x \text{ and } y *^e x = y.$$

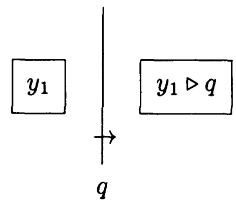

Figure 4:

- For any $y \in Y$, $x_1, x_2 \in X$, and any $g, h \in G$,

$$(y *^g x_1) *^h x_2 = (y *^h x_2) *^{h^{-1}gh} (x_1 *^h x_2).$$

Put $y \triangleright (x, g) := y *^g x$ for $y \in Y$, $(x, g) \in Q$. Then the second axiom implies that $(y \triangleright q_1) \triangleright q_2 = (y \triangleright q_2) \triangleright (q_1 \triangleright q_2)$ for $q_1, q_2 \in Q$. Any G-family of quandles $(X, \{*^g\}_{g \in G})$ itself is an X-set with its binary operations. Any singleton set $\{y\}$ is also an X-set with the maps $*^g$ defined by $y *^g x = y$ for $x \in X$ and $g \in G$, which is a trivial X-set.

Let D be a diagram of an oriented spatial trivalent graph. We denote by $\mathcal{R}(D)$ the set of complementary regions of D. Let X be a G-family of quandles, and Y an X-set. Let Q be the associated quandle of X. An *X_Y-coloring* of D is a map $C : \mathcal{A}(D) \cup \mathcal{R}(D) \to Q \cup Y$ satisfying the following conditions.

- $C(\mathcal{A}(D)) \subset Q$, $C(\mathcal{R}(D)) \subset Y$.
- The restriction $C|_{\mathcal{A}(D)}$ of C on $\mathcal{A}(D)$ is an X-coloring of D.
- For any arc $\alpha \in \mathcal{A}(D)$, we have

$$C(\alpha_1) \triangleright C(\alpha) = C(\alpha_2),$$

where α_1, α_2 are the regions facing the arc α so that the normal orientation of α points from α_1 to α_2 (see Figure 4).

We denote by $\mathrm{Col}_X(D)_Y$ the set of X_Y-colorings of D.

For two diagrams D and E which locally differ, we denote by $\mathcal{R}(D, E)$ the set of regions that D and E share.

Lemma 3.3. *Let X be a G-family of quandles, Y an X-set. Let D be a diagram of an oriented spatial trivalent graph. Let E be a diagram obtained by applying one of the R1–R6 moves to the diagram D once, where we choose orientations for E which agree with those for D on $\mathcal{A}(D, E)$. For $C \in \mathrm{Col}_X(D)_Y$, there is a unique X_Y-coloring $C_{D,E} \in \mathrm{Col}_X(E)_Y$ such that $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$ and $C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{R}(D,E)}$.*

4 A homology

Let X be a G-family of quandles, and Y an X-set. Let $(Q, \triangleright)$ be the associated quandle of X. Let $B_n(X)_Y$ be the free abelian group generated by the elements of $Y \times Q^n$ if $n \geq 0$, and let $B_n(X)_Y = 0$ otherwise. We put

$$((y, q_1, \dots, q_i) \triangleright q, q_{i+1}, \dots, q_n) := (y \triangleright q, q_1 \triangleright q, \dots, q_i \triangleright q, q_{i+1}, \dots, q_n)$$

for $y \in Y$ and $q, q_1 \dots, q_n \in Q$. We define a boundary homomorphism $\partial_n : B_n(X)_Y \to B_{n-1}(X)_Y$ by

$$\begin{aligned}\partial_n(y, q_1, \dots, q_n) = &\sum_{i=1}^{n} (-1)^i (y, q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n) \\ &- \sum_{i=1}^{n} (-1)^i ((y, q_1, \dots, q_{i-1}) \triangleright q_i, q_{i+1}, \dots, q_n)\end{aligned}$$

for $n > 0$, and $\partial_n = 0$ otherwise. Then $B_*(X)_Y = (B_n(X)_Y, \partial_n)$ is a chain complex (see [1, 2, 4, 5]).

Let $D_n(X)_Y$ be the subgroup of $B_n(X)_Y$ generated by the elements of

$$\bigcup_{i=1}^{n-1} \left\{ (y, q_1, \dots, q_{i-1}, (x, g), (x, h), q_{i+2}, \dots, q_n) \;\middle|\; \begin{array}{l} y \in Y,\ x \in X,\ g, h \in G \\ q_1, \dots, q_n \in Q \end{array} \right\}$$

and

$$\bigcup_{i=1}^{n} \left\{ \begin{array}{l} (y, q_1, \dots, q_{i-1}, (x, gh), q_{i+1}, \dots, q_n) \\ -(y, q_1, \dots, q_{i-1}, (x, g), q_{i+1}, \dots, q_n) \\ -((y, q_1, \dots, q_{i-1}) \triangleright (x, g), (x, h), q_{i+1}, \dots, q_n) \end{array} \;\middle|\; \begin{array}{l} y \in Y,\ x \in X, \\ g, h \in G, \\ q_1, \dots, q_n \in Q \end{array} \right\}.$$

We remark that

$$(y, q_1, \dots, q_{i-1}, (x, e), q_{i+1}, \dots, q_n)$$

and

$$\begin{aligned}&(y, q_1, \dots, q_{i-1}, (x, g), q_{i+1}, \dots, q_n) \\ &+ ((y, q_1, \dots, q_{i-1}) \triangleright (x, g), (x, g^{-1}), q_{i+1}, \dots, q_n)\end{aligned}$$

belong to $D_n(X)_Y$.

Lemma 4.1. *For $n \in \mathbb{Z}$, we have $\partial_n(D_n(X)_Y) \subset D_{n-1}(X)_Y$. Thus $D_*(X)_Y = (D_n(X)_Y, \partial_n)$ is a subcomplex of $B_*(X)_Y$.*

We put $C_n(X)_Y = B_n(X)_Y / D_n(X)_Y$. Then $C_*(X)_Y = (C_n(X)_Y, \partial_n)$ is a chain complex. For an abelian group A, we define the cochain complex $C^*(X; A)_Y = \mathrm{Hom}(C_*(X)_Y, A)$. We denote by $H_n(X)_Y$ the nth homology group of $C_*(X)_Y$.

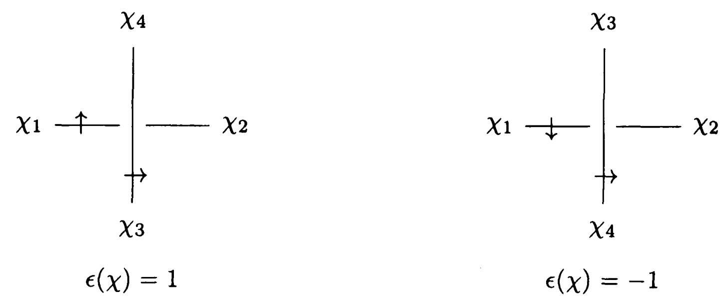

Figure 5:

5 Cocycle invariants

Let X be a G-family of quandles, and Y an X-set. Let D be a diagram of an oriented spatial trivalent graph. For an X_Y-coloring $C \in \mathrm{Col}_X(D)_Y$, we define the weight $w(\chi; C) \in C_2(X)_Y$ at a crossing χ of D as follows. Let χ_1, χ_2 and χ_3 be respectively the under-arcs and the over-arc at a crossing χ such that the normal orientation of χ_3 points from χ_1 to χ_2. Let R_χ be the region facing χ_1 and χ_3 such that the normal orientations χ_1 and χ_3 point from R_χ to the opposite regions with respect to χ_1 and χ_3, respectively. Then we define

$$w(\chi; C) = \epsilon(\chi)(C(R_\chi), C(\chi_1), C(\chi_3)),$$

where $\epsilon(\chi) \in \{1, -1\}$ is the sign of a crossing χ. We define a chain $W(D; C) \in C_2(X)_Y$ by

$$W(D; C) = \sum_\chi w(\chi; C),$$

where χ runs over all crossings of D.

Lemma 5.1. *The chain $W(D; C)$ is a 2-cycle of $C_*(X)_Y$. Further, for cohomologous 2-cocycles θ, θ' of $C^*(X; A)_Y$, we have $\theta(W(D; C)) = \theta'(W(D; C))$.*

Lemma 5.2. *Let D be a diagram of an oriented spatial trivalent graph. Let E be a diagram obtained by applying one of the R1–R6 moves to the diagram D once, where we choose orientations for E which agree with those for D on $\mathcal{A}(D, E)$. For $C \in \mathrm{Col}_X(D)_Y$ and $C_{D,E} \in \mathrm{Col}_X(E)_Y$ such that $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$ and $C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{R}(D,E)}$, we have $[W(D; C)] = [W(E; C_{D,E})] \in H_2(X)_Y$.*

We denote by G_H (resp. G_K) the fundamental group of the exterior of a handlebody-link H (resp. a spatial graph K). When H is represented by K, the groups G_H and G_K are identical. Let D be a diagram of an oriented spatial trivalent graph K. By the definition of an X_Y-coloring C of D, the map $p_G \circ C|_{A(D)}$ represents a homomorphism from G_K to G, which we denote by $\rho_C \in \mathrm{Hom}(G_K, G)$. For $\rho \in \mathrm{Hom}(G_K, G)$, we define

$$\mathrm{Col}_X(D;\rho)_Y = \{C \in \mathrm{Col}_X(D)_Y \mid \rho_C = \rho\}.$$

For a 2-cocycle θ of $C^*(X;A)_Y$, we define

$$\begin{aligned}
\mathcal{H}(D) &:= \{[W(D;C)] \in H_2(X)_Y \mid C \in \mathrm{Col}_X(D)_Y\},\\
\Phi_\theta(D) &:= \{\theta(W(D;C)) \in A \mid C \in \mathrm{Col}_X(D)_Y\},\\
\mathcal{H}(D;\rho) &:= \{[W(D;C)] \in H_2(X)_Y \mid C \in \mathrm{Col}_X(D;\rho)_Y\},\\
\Phi_\theta(D;\rho) &:= \{\theta(W(D;C)) \in A \mid C \in \mathrm{Col}_X(D;\rho)_Y\}
\end{aligned}$$

as multisets.

Lemma 5.3. *Let D be a diagram of an oriented spatial trivalent graph K. For $\rho, \rho' \in \mathrm{Hom}(G_K, G)$ such that ρ and ρ' are conjugate, we have $\mathcal{H}(D;\rho) = \mathcal{H}(D;\rho')$ and $\Phi_\theta(D;\rho) = \Phi_\theta(D;\rho')$.*

We denote by $\mathrm{Conj}(G_K, G)$ the set of conjugacy classes of homomorphisms from G_K to G. By Lemma 5.3, $\mathcal{H}(D;\rho)$ and $\Phi_\theta(D;\rho)$ are well-defined for $\rho \in \mathrm{Conj}(G_K, G)$.

Lemma 5.4. *Let D be a diagram of an oriented spatial trivalent graph K. Let E be a diagram obtained from D by reversing the orientation of an edge e. For $\rho \in \mathrm{Hom}(G_K, G)$, we have $\mathcal{H}(D) = \mathcal{H}(E)$, $\Phi_\theta(D) = \Phi_\theta(E)$, $\mathcal{H}(D;\rho) = \mathcal{H}(E;\rho)$ and $\Phi_\theta(D;\rho) = \Phi_\theta(E;\rho)$.*

By Lemma 5.4, $\mathcal{H}(D)$, $\Phi_\theta(D)$, $\mathcal{H}(D;\rho)$ and $\Phi_\theta(D;\rho)$ are well-defined for a diagram D of an unoriented spatial trivalent graph, which is a diagram of a handlebody-link. For a diagram D of a handlebody-link H, we define

$$\begin{aligned}
\mathcal{H}^{\mathrm{hom}}(D) &:= \{\mathcal{H}(D;\rho) \mid \rho \in \mathrm{Hom}(G_H, G)\},\\
\Phi_\theta^{\mathrm{hom}}(D) &:= \{\Phi_\theta(D;\rho) \mid \rho \in \mathrm{Hom}(G_H, G)\},\\
\mathcal{H}^{\mathrm{conj}}(D) &:= \{\mathcal{H}(D;\rho) \mid \rho \in \mathrm{Conj}(G_H, G)\},\\
\Phi_\theta^{\mathrm{conj}}(D) &:= \{\Phi_\theta(D;\rho) \mid \rho \in \mathrm{Conj}(G_H, G)\}
\end{aligned}$$

as "multisets of multisets". We remark that, for X_Y-colorings C and $C_{D,E}$ in Lemma 5.2, we have $\rho_C = \rho_{C_{D,E}}$. Then, by Lemmas 5.1–5.4, we have the following theorem.

Theorem 5.5. *Let X be a G-family of quandles, Y an X-set. Let θ be a 2-cocycle of $C^*(X;A)_Y$. Let H be a handlebody-link represented by a diagram D. Then the following are invariants of a handlebody-link H.*

$$\mathcal{H}(D), \quad \Phi_\theta(D), \quad \mathcal{H}^{\mathrm{hom}}(D), \quad \Phi_\theta^{\mathrm{hom}}(D), \quad \mathcal{H}^{\mathrm{conj}}(D), \quad \Phi_\theta^{\mathrm{conj}}(D).$$

We denote the invariants of H given in Theorem 5.5 by

$$\mathcal{H}(H), \quad \Phi_\theta(H), \quad \mathcal{H}^{\mathrm{hom}}(H), \quad \Phi_\theta^{\mathrm{hom}}(H), \quad \mathcal{H}^{\mathrm{conj}}(H), \quad \Phi_\theta^{\mathrm{conj}}(H),$$

respectively.

We denote by H^* the mirror image of a handlebody-link H. Then we have the following theorem.

Theorem 5.6. *For a handlebody-link H, we have*

$$\begin{aligned} \mathcal{H}(H^*) &= -\mathcal{H}(H), & \Phi_\theta(H^*) &= -\Phi_\theta(H), \\ \mathcal{H}^{\mathrm{hom}}(H^*) &= -\mathcal{H}^{\mathrm{hom}}(H), & \Phi_\theta^{\mathrm{hom}}(H^*) &= -\Phi_\theta^{\mathrm{hom}}(H), \\ \mathcal{H}^{\mathrm{conj}}(H^*) &= -\mathcal{H}^{\mathrm{conj}}(H), & \Phi_\theta^{\mathrm{conj}}(H^*) &= -\Phi_\theta^{\mathrm{conj}}(H), \end{aligned}$$

where $-S = \{-a \mid a \in S\}$ for a multiset S.

6 Applications

In this section, we calculate cocycle invariants defined in the previous section for the handlebody-knots $0_1, \ldots, 6_{16}$ in the table given in [9], by using a 2-cocycle given by Nosaka [18]. This calculation enables us to distinguish some of handlebody-knots from their mirror images, and a pair of handlebody-knots whose complements have isomorphic fundamental groups.

Let $G = SL(2;\mathbb{Z}_3)$ and $X = (\mathbb{Z}_3)^2$. Then X is a G-family of quandles with the proper binary operation as given in Proposition 2.2 (2). Let Y be the trivial X-set $\{y\}$. We define a map $\theta : Y \times (X \times G)^2 \to \mathbb{Z}_3$ by

$$\theta(y,(x_1,g_1),(x_2,g_2)) := \lambda(g_1)\det(x_1 - x_2, x_2(1 - g_2^{-1})),$$

where the abelianization $\lambda : G \to \mathbb{Z}_3$ is given by

$$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+d)(b-c)(1-bc).$$

	$\Phi_\theta(H)$
0_1	$\{\{0_9\}_{76}\}$
4_1	$\{\{0_9\}_{83},\{0_{27}\}_{22},\{0_{81}\}_3\}$
5_1	$\{\{0_9\}_{76}\}$
5_2	$\{\{0_9\}_{95},\{0_{27}\}_6,\{0_{81}\}_1,\{0_9,1_{18}\}_4,\{0_{27},1_{54}\}_2\}$
5_3	$\{\{0_9\}_{102},\{0_{27}\}_4,\{0_{27},2_{54}\}_2\}$
5_4	$\{\{0_9\}_{74},\{0_{81}\}_2\}$
6_1	$\{\{0_9\}_{91},\{0_{27}\}_{16},\{0_{81}\}_1\}$
6_2	$\{\{0_9\}_{106},\{0_{45},1_{18},2_{18}\}_2\}$
6_3	$\{\{0_9\}_{74},\{0_{27}\}_2\}$
6_4	$\{\{0_9\}_{76}\}$
6_5	$\{\{0_9\}_{74},\{0_9,1_{18}\}_2\}$
6_6	$\{\{0_9\}_{72},\{0_{27}\}_4\}$
6_7	$\{\{0_9\}_{85},\{0_{27}\}_{16},\{0_{81}\}_3,\{0_{45},1_{18},2_{18}\}_4\}$
6_8	$\{\{0_9\}_{76}\}$
6_9	$\{\{0_9\}_{91},\{0_{27}\}_6,\{0_{81}\}_1,\{0_9,1_{18}\}_6,\{0_{27},1_{54}\}_2,\{0_{27},2_{54}\}_2\}$
6_{10}	$\{\{0_9\}_{76}\}$
6_{11}	$\{\{0_9\}_{70},\{0_9,1_{18}\}_6\}$
6_{12}	$\{\{0_9\}_{97},\{0_{81}\}_1,\{0_9,1_{18}\}_8,\{0_9,1_{36},2_{36}\}_2\}$
6_{13}	$\{\{0_9\}_{95},\{0_{27}\}_6,\{0_{81}\}_1,\{0_9,2_{18}\}_4,\{0_{27},2_{54}\}_2\}$
6_{14}	$\{\{0_9\}_{119},\{0_{27}\}_6,\{0_{81}\}_{11},\{0_9,1_{18}\}_{12},\{0_{27},1_{54}\}_{24}\}$
6_{15}	$\{\{0_9\}_{119},\{0_{27}\}_6,\{0_{81}\}_{11},\{0_9,2_{18}\}_{12},\{0_{27},1_{54}\}_{24}\}$
6_{16}	$\{\{0_9\}_{44},\{0_{81}\}_{32}\}$

表 1:

By [18], the map θ is a 2-cocycle of $C^*(X;\mathbb{Z}_3)_Y$. Table 1 lists the invariant $\Phi_\theta^{\mathrm{conj}}(H)$ for the handlebody-knots $0_1,\ldots,6_{16}$. We represent the multiplicity of elements of a multiset by using subscripts. For example, $\{\{0_2,1_3\}_1,\{0_3\}_2\}$ represents the multiset $\{\{0,0,1,1,1\},\{0,0,0\},\{0,0,0\}\}$.

From Table 1, we see that our invariant can distinguish the handlebody-knots 6_{14}, 6_{15}, whose complements have the isomorphic fundamental groups. Together with Theorem 5.6, we also see that handlebody-knots 5_2, 5_3, 6_5, 6_9, 6_{11}, 6_{12}, 6_{13}, 6_{14}, 6_{15} are not equivalent to their mirror images. In particular, the chiralities of 5_3, 6_5, 6_{11} and 6_{12} were not known. Table 2 shows us known facts on the chirality of handlebody-knots in [9] so far. In the column of "chirality", the symbols $\bigcirc$ and $\times$ mean that the handlebody-knot is amphichiral and chiral, respectively, and the symbol ? means that it is not known whether the handlebody-knot is amphichiral or chiral. The symbols ✓ in the right five columns mean that the handlebody-knots can be proved chiral by using the method introduced

	chirality	M	II	LL	IKO	IIJO
0_1	○					
4_1	○					
5_1	×			✓		
5_2	×		✓	✓		✓
5_3	×					✓
5_4	×				✓	
6_1	×	✓				
6_2	?					
6_3	?					
6_4	×			✓		
6_5	×					✓
6_6	○					
6_7	○					
6_8	?					
6_9	×		✓			✓
6_{10}	?					
6_{11}	×					✓
6_{12}	×					✓
6_{13}	×		✓	✓		✓
6_{14}	×				✓	✓
6_{15}	×				✓	✓
6_{16}	○					

表 2:

in the papers corresponding to the columns. Here, M, II, LL, IKO and IIJO denote the papers [17], [7], [15], [10] and this paper, respectively.

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